REMARKS ON THE PROGRESS OF CELESTIAL MECHANICS SINCE THE MIDDLE OF THE CENTURY.

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THE application of mathematics to the solution of the problems presented by the motion of the heavenly bodies has had a larger degree of success than the same application in the case of the other departments of physics. This is probably due to two causes. The principal objects to be treated in the former case are visible every clear night, consequently the questions connected with them received earlier attention; while, in the latter case, the phenomena to be discussed must ofttimes be produced by artificial means in the laboratory; and the discovery of certain classes of them, as, for instance, the property of magnetism, may justly be attributed to accident. A second cause is undoubtedly to be found in the fact that the application of quantitative reasoning to what is usually denominated as physics generally leads to a more difficult department of mathematics than in the case of the motion of the heavenly bodies. In the latter we have but one independent variable, the time; while in the former generally several are present, which makes the difference of having to integrate ordinary differential equations or those which are partial. Thus it happens that, while the science of astro-mechanics is started by Newton, that of thermal conductivity receives its first treatment, at the hands of Fourier, more than a century later. In addition to these two causes, ever since the discovery of the telescope the application of optical means to the discovery of whatever might be found in the heavens has always had a fascination for mankind. And, as the ability to co-ordinate and correlate the facts observed much enhances the enjoyment of scientific occupation, it has resulted that many who began as observers ended as mathematical astronomers. Thus our science has had relatively a large number of cultivators.

A thoroughly satisfactory history of our subject is yet to be written. We have only either slight sketches of the whole, or elaborate treatments of special divisions of the science, and none of them coming down to recent times. Among the former may be mentioned Gautier's *Essai historique sur le pro-*

blème des trois corps, which appeared in 1817. Also Laplace's historical chapters in the last volume of the *Mécanique Céleste*. Todhunter's History of the theories of attraction and the figure of the earth is an example of the latter class. Such books as Todhunter's — of which Delambre has given an earlier example in his *Histoire de l'Astronomie* — can hardly be regarded as history; they resemble rather extensive tables of contents of the literature examined, accompanied by short comments. However, in many cases, they are more useful to the student than formal histories would be, as, when judiciously compiled, they may, as epitomes in our libraries, take the place of a large mass of scientific literature. The History of Physical Astronomy, by Robert Grant, is a book that comes down to 1850, and professedly covers the whole of our subject. But only one third of this book is devoted to astro-mechanics, the rest dealing with what is really observational and descriptive astronomy. Moreover, the author indulges so much in diffusive veins of writing, that but a small fraction of the 200 pages is really given to purely historic statement. As far as the Lunar Theory is concerned, the third volume of M. Tisserand's *Traité de Mécanique Céleste* constitutes a fair history. But it must be borne in mind that the author's plan is to notice only the disquisitions having a first-class importance; hence his history is incomplete in this respect.

In America we are not well situated for investigations of this character, on account of the meagreness of our libraries. Of no inconsiderable number of memoirs and even books, having at least some importance in our subject, there exist no copies in the United States. Hence, should an American be inclined to undertake the task of writing the history of our subject, he must at least perform some of the work abroad.

In the present discourse it is proposed to touch very lightly the more important steps made since the middle of the century, the time at our disposal not admitting fuller treatment.

And first we will take up Delaunay's method, proposed for employment in the lunar theory, but quite readily extended to all classes of problems in dynamics. The first sketch of this method, given of course by the author himself, appeared in the *Comptes Rendus* of the Paris Academy of Sciences, in 1846. It professes to be merely an extract from a memoir offered for publication in the collections of the Academy, which must, however, have been afterwards withdrawn to make place for the two volumes of the *Théorie du Mouvement de la Lune*. When this extract is compared with the earlier chapters of the latter work, it is perceived that Delaunay has, to some extent, modified and improved his method in the interim between 1846 and 1860. In this long period nothing appeared from the author on this subject. He must have been profoundly engaged in

applying his method to the motion of the moon. Tisserand's exposition of this method is somewhat more brief than the author's own. But when the necessary modifications are introduced into Delaunay's procedures, to make them applicable to the more general case of the motion of a system of bodies, the establishment of the formulas can be rendered still more brief.

There is one point in reference to Delaunay's method which, as far as I am aware, has escaped notice. This method consists in a series of operations or transformations, in each of which the position of the moon in space is defined by six variables, the number three being doubled in order that the velocities, as well as the co-ordinates, may be expressed without differentials. The aim of the transformations is to make one half of these, which Poincaré has called the linear variables, continually approach constancy, while the other half, named the angular variables, continually approach a linear function of the time. But at any stage of the process the position of the moon, as well as its velocity, is definitely fixed by the six variables produced by the last transformation, provided that the proper degree of variability is attributed to them, just as, before any transformation was made, the six elements of elliptic motion, usually denominated osculating, defined them; the point of difference to be noticed being that the more the transformations are multiplied, the more complex becomes the character of the expression of the former quantities in terms of the latter. But, however great may be the number of transformations, the series evolved have always one consistent trait, viz., that the angular variables are involved in them only through cosines or sines of linear functions of these variables, the linear functions being formed with integral coefficients. Now, as in all this work we are obliged to employ infinite series, the question of their convergence is an extremely important one. The inquiry in this respect may be divided into two parts, mainly independent of each other. These are, convergence as respects the angular variables, and convergence as respects the linear variables. The first part is much the more simple. Regarding each of the coefficients of the series we employ as a whole, that is, representing it by a definite integral, it is quite easily perceived that the said series are both legitimate and convergent when, giving the angular variables the utmost range of values, still no two of the bodies can occupy the same point of space. In the contrary case the series are evidently divergent. This condition affords certain limiting conditions for the values of the linear variables. Could we trace these limiting conditions through all the transformations, and obtain by comparison the formulas to which these tend when the number of transformations is made infinite, we should be in possession of the conditions of stability of motion of the system of bodies. The second

part of the inquiry relates to the expression of the mentioned coefficients by infinite series proceeding according to powers and products of certain parameters which are functions of the linear variables. It is well known that, in the case of elliptic elements, Laplace and Cauchy almost simultaneously showed that the series are convergent when the eccentricity does not exceed a fraction which is about two-thirds. The determination of the conditions of convergence, after certain transformations have been made in the signification of the elements, is undoubtedly a more complex problem; nevertheless, it seems to be within the competency of analysis as it exists at present.

The discovery of the criterion for the convergence of series proceeding according to powers and products of parameters is due to Cauchy, and is a most remarkable contribution to the science of mathematics. Supposing that the parameters begin from zero values, this criterion amounts to saying that the moment the function, which the series is to represent, ceases to be holomorphic, or becomes infinite, that moment the series ceases to be convergent. Consequently, if a space, having as many dimensions as there are parameters in the case, be conceived, and a surface be constructed in it formed by the consensus of all the points where the considered function ceases to be holomorphic, then, provided the values of the parameters define a point within this surface, that is, on the same side where lies the origin, the series will be convergent. Generally this surface will be closed, and, within it, the function will not take infinity as its value.

Without any mathematical reasoning the propriety of the principle just enunciated may be perceived. Since it is possible for the series in powers and products to give only one value for the function, the moment the latter may have any one of several values, the series fails to give them all; and, as there is no reason why any particular value should be selected, the conclusion must be that it does not represent any of them. Also, it is easy to see that, when the function takes infinity as its value, the series fails to represent it.

In applying this principle to the series involved in the treatment of the problem of many bodies by Delaunay's method, it appears, at first sight, as if we must have some finite representation of the coefficients in question in order to discover the particular points at which they cease to be holomorphic, such, for instance, as is given by an algebraic or transcendental equation. But this is not imperative, as it is often possible to make this discovery from certain recognized properties of the function considered, without being in possession of its form explicitly or implicitly. It appears probable that, in the class of cases considered, the mentioned coefficients can be represented by multiple definite integrals, all taken between the limits 0

and π, the independent variables being those which have been denominated angular. Such functions are always holomorphic, provided that the expressions under the signs of integration are themselves holomorphic between the mentioned limits. If the statement just made be admitted, although it may be impossible to write explicitly the mentioned expressions, we may, nevertheless, be certain that they remain holomorphic, provided that the linear variables, which may be the same as the parameters considered, are so restricted in their range of values that no matter what values the angular variables receive, no distance between any two bodies of the system can vanish. Or, in other words, that the R of Delaunay must never become infinite. Thus it seems probable that the conditions of convergence for Delaunay's series are precisely identical with those for the stability of motion of the system.

The series arising in Delaunay's method, as applied to the moon, contain five parameters; the number would be six were the moon's mass not neglected. We should also have six in the application of the method to two planets moving about the sun; however, should we employ the well-known functions $b_s^{(i)}$ of Laplace, the number would be reduced to five. It ought to be possible, therefore, after the performance of a limited number of operations, to assign limiting values to these parameters, below which the series would certainly be convergent. This also involves the possibility of finding limits to the errors committed by truncating the series at a certain order of terms. Again, provided the time is limited to a certain interval, the capacity of these truncated series for representing the co-ordinates of the planets could be shown by giving superior limits to the errors necessarily involved.

One more remark may be made before we leave Delaunay's method. In every operation or transformation half the integrals are obtained without the intervention of the time, and from these solely are obtained the ranges of values for all the linear variables. As no integrating divisors appear in their expressions, it follows that the question of stability is not affected in any way by the vanishing of these. Moreover, the presence of a libration in the angle of operation does not necessitate any change in the procedure. The integrating divisors which appear in the expressions for the angular variables, obtained through quadratures, may cause difficulty, but this can generally be removed by a modification of the parameters employed in the development of the coefficients in series. Beyond this it does not seem necessary to attend particularly to the terms which Professor Gyldén has designated as critical.

To give a succinct idea of the scope of this method, it may be said that it is applicable whenever, in the system, the planets maintain their order of succession from the sun. In systems

where that undergoes change, as is the case with the group of minor planets, supposing their action on each other is sensible, it is not applicable.

Delaunay's method has not yet received all the developments and applications it is susceptible of.

The treatise of Hansen on the shortest and most ready method of deriving the perturbations of the small planets was published in the interval 1857–1861. But as the principles on which it is founded had been elaborated and communicated to the public some years earlier, it is, perhaps, more properly to be assigned to the first half of the century. In consequence, I pass it over with this slight mention.

Perhaps the most conspicuous labors in our subject, during the period of time we consider, are those of Professor Gyldén and M. Poincaré. We will limit our attention, for the remainder of this discourse, to the consideration of these investigations.

Professor Gyldén began work with the methods of Hansen and was gradually led to modifications of them looking towards their use for indefinite lengths of time. This quality has latterly become imperative with him, and he has recently published the first volume of what is evidently intended to be a lengthy work entitled *Traité Analytique des Orbites Absolues des Huit Planètes Principales.* To show the drift of Professor Gyldén's investigations, we cannot do better than give an analysis of this volume. At the outset the author introduces a class of curves he names periphlegmatic, that is, curves which surround a flame. The definition of this sort of curve is that it describes continually the space between two concentric spheres, and, at every point, turns its concavity towards the intersection of the radius vector with the inner sphere. In an application to the solar system, the sun is supposed to occupy the common centre of the spheres. The investigation is at first limited to the case where this curve is plane. A differential equation of the second order is derived which the radius vector of this curve satisfies, the independent variable being the angle described. The perpendicular distance between the spheres is called the diastem. The spheres are supposed to be drawn so that they touch the curve at the points where the radius becomes a maximum or minimum. Thus, in some cases, the spheres are regarded as fixed, in others, as movable. In the latter case, however, the sum of their radii is supposed to remain constant. Thence we have two groups of periphlegmatic curves; those with constant and those with variable diastems. The author gives examples of both these groups, in most cases of which the line of apsides is variable, and considers the situation and density of the points of intersection of these curves with themselves.

The idea of an absolute orbit of a planetary body is this: an oval symmetrical with regard to an axis movable in space. While the axis remains constant in length (the half of it is called the protometre), the velocity of its motion may vary, and the diastem may also vary. Professor Gyldén, however, admits into the expressions of these variations only terms whose period would become infinite did the planetary masses vanish. These terms he calls elementary. But elementary terms in the diastem and the longitude of the perihelion can produce terms in the co-ordinates having periods which differ but little from the time of revolution of the planet. These are also called elementary terms. But the two classes are distinguished, the first as being of the type (A), and the second as of the type (B). In all the formulas relative to this matter the author insists on keeping the arc described by the radius as the independent variable.

The co-ordinates are only approximately given by the preceding apparatus of expressions. They must then have certain complements added to them; these, however, are all composed of terms which would vanish with the planetary masses.

In deriving the elementary terms in the radius of a planet through the integration of a linear differential equation of the second order, Professor Gyldén attaches much price to his method of establishing the convergence of the series formed by the successive terms. As the latter are obtained through division by divisors of the order of the planetary masses, it might be feared that some of them would turn out to be very large. But the author prevents this by retaining in the coefficient of the dependent variable in the differential equation a quantity equivalent to the sum of the squares of all the coefficients in the integral. This is named the horistic or limiting function. It is plain such an expression could be introduced in the mentioned coefficient, provided that the linear equation is the truncated form of an equation containing the cube of the variable. And in the problem of planetary motion the approximations may always be so ordered that this shall be the case.

With regard to the co-ordinate which exhibits the departure of the planet from a fixed plane, Professor Gyldén does not greatly deviate from the procedure of Hansen in following the displacement of the instantaneous plane of the orbit. Only here, as in the preceding treatment of the radius, he would sharply distinguish the elementary and non-elementary terms.

At this point is introduced certain new nomenclature. As before we had diastem now we have anastem to denote the product of the radius and the sine of the inclination; and what has generally been called the true argument of the latitude is here called the anastematic argument. Any angular magnitudes which are constantly moving through the circumference are

astronomic arguments; and when they have the same mean velocity of rotation they are isokinetic; and isokinetic arguments are homorhythmic when, in each revolution through the circumference, they always retake together the same corresponding points. In like manner, the true anomaly is the diastematic argument, and we have diastematic and anastematic coefficients and moduli. It will be seen from this that Professor Gyldén does not shrink from imposing on us the labor of learning new terms.

Thus far we have been engaged in deriving the equations of the path followed by a heavenly body; it remains to show how we may find the point on that path occupied by the body at a given moment. There is then necessary an equation between the time and the variable assumed as independent, that is, the orbit longitude, or, more properly, the amount of angle described by the radius vector. If we suppose the absolute orbit to be described by the planet so that equal areas are passed over by the radius in equal times, it is plain that, on the attainment of a given longitude, a definite amount of time must have elapsed since the epoch. This is what Professor Gyldén calls the *reduced* time; and he computes the difference between it and the actual time required by the theory of gravity for the planet to arrive at the stated direction. This mode of proceeding does not differ from Hansen's except in the point that the absolute orbit is substituted for a fixed ellipse.

But this gives us correctly only the orbit longitude; for the radius and the latitude, which correspond in the absolute orbit to this reduced time, are not quite those which the planet has at the actual time. Consequently, Professor Gyldén proposes to compute two corrections, the one to be applied to the product of the eccentricity into the cosine of the true anomaly, the other to the sine of the latitude. Also the reduction of the orbit longitude to the plane of reference must be manipulated so that it comes out correctly.

The employment of the orbit longitude as independent variable throughout all the integrations necessitates a mass of very intricate transformations of terms from one shape into another. Also the integrations which bear on elementary terms must be kept distinct from those which bear on non-elementary terms. A degree of complexity is thus imparted to the subject, which makes it difficult to see when one has really gathered up all the warp and woof of it. Professor Gyldén has nowhere removed the scaffolding from the front of his building and allowed us to see what architectural beauty it may possess; it is necessary to compare a large number of equations scattered through the volume before one can opine how the author means to proceed.

The advantages claimed for the method are that it prevents

the time from appearing outside the trigonometrical functions, and that it escapes all criticism on the score of convergence. The first is readily conceded, but many simpler methods possessing this advantage are already elaborated, and it is not so clear that the second ought to be granted.

No completely worked out example of the application of this method has yet been published. The great labor involved will naturally deter investigators from employing it.

In 1890 was published the memoir of M. H. Poincaré, entitled *Sur le problème des trois corps et les équations de la dynamique*, and which obtained the prize of the King of Sweden. Most of the results of this memoir were worked over and presented anew with greater elaboration and clearness by their author in *Les Méthodes Nouvelles de la Mécanique Céleste*. Here we find a large number of new and very interesting theorems.

First is to be noted the class of particular solutions in the problem of the motion of a system of material points which are now named *periodic solutions*. The initial relative positions and velocities of the several points are so adjusted that, after the lapse of a definite time, the latter retake them. Hence is evident a method which may be employed to elaborate this special case of motion, viz., by the tentative process with mechanical quadratures. M. Poincaré has divided this sort of solutions into three classes, of which, however, the second and third are not essentially different. He has shown that, in the latter classes, the values of the arbitrary constants of the problem must be so adjusted that no secular inequalities, or, as Professor Gyldén calls them, elementary terms, may arise. The number and variety of these particular solutions is far greater than one would at first sight imagine.

We come now to a second class of particular solutions named by the author *asymptotic*. It arises from the consideration of solutions differing very little from periodic solutions. Here we have to deal with linear differential equations having periodic coefficients. The integrals of these contain in their terms exponential factors, and on the nature of the exponents of these factors depends the quality of the resulting solutions. M. Poincaré has named these exponents *characteristic*. They are roots of an algebraic equation of a degree equal to the number of dependent variables involved in the question. If any of these roots are imaginary with real portions or wholly real, we are in presence of asymptotic solutions. The algebraic equation mentioned contains the unknown only in even powers; hence the characteristic exponents are in pairs having the same absolute value, but with contrary signs. In all the cases presented by astronomy, where, on account of the near approach to circular motion, a periodic solution can be taken as

a first approximation, it appears that the squares of the characteristic exponents are all real and negative. Thus, there is no call here to consider this sort of solution, and this fact must much diminish the interest of the astronomer in it. M. Poincaré has, however, elaborated it with great pains, showing how the effect of higher powers of the deviations from the periodic solution may be taken into account. The series resulting are, nevertheless, divergent, as in other cases.

The second volume of the *Méthodes Nouvelles* is devoted to the elaboration and consideration of various processes for developing the integrals of planetary motion according to the powers of a small parameter. The chief of these are due to Professor Newcomb and MM. Lindstedt and Bohlin; but M. Poincaré has augmented the number of them by introducing modifications of his own. All involve the principle of recurrence; that is, the first step is the only one which is independent, the following depend on all that precede. These methods, in their general aspect, do not differ from the old developments in powers of the disturbing force, except the operations are so adjusted that the time never escapes from the trigonometric functions. This is accomplished by greatly augmenting the number of the elementary arguments, and by supposing that the rate of motion of each of these is developable according to integral powers of the before-mentioned parameter, or, in some cases, of its square root.

When there is more than one elementary argument, the series obtained in all these ways are pronounced to be generally divergent in the rigorous sense of the word. M. Poincaré brings forward several methods of proof of this. The first depends on the presence of small divisors in the expressions of the coefficients. However, when we do not insist on developments in powers of a parameter, this method of proof has no application. Another method is derived from the principle that two characteristic exponents vanish for every uniform integral that exists. But the integrals which necessitate this conclusion must not only be uniform, they must be valid for every possible case of the problem. Now the integrals known as those of the conservation of living forces and of areas are of this nature; but the integrals derivable from the series of Delaunay, Newcomb, and Lindstedt are valid only for a limited range in the values of the linear variables. For instance, in the problem of the three bodies, if the deformation of the triangle formed by these bodies is such that we cannot find any two sides, one of which sustains to the other an invariable relation of greater to less, we cannot apply the mentioned series. And here it is well to note that the defect of convergence does not arise from the application of the processes

of integration, but already exists in the development of the perturbative function before integration commences. Thus Delaunay's development of this function at the beginning of his lunar theory is divergent and illusory, unless we have the lunar radius in apogee always less than the solar radius in perigee, and that without regard to the mode of expressing the coefficients. Some of the particular integrals, relied upon by M. Poincaré to establish the vanishing of all the characteristic exponents in case we accept M. Lindstedt's series as valid, lie, so to speak, on the boundary of the domain in which these series are convergent.

In the third place an appeal is made to the alleged non-existence of analytic and uniform integrals beyond those already known. Were this non-existence clearly established it would decide the question on the side where M. Poincaré has placed himself. But, at least as far as the non-existence of integrals of this nature in a limited domain for the linear variables is concerned, the proof given for it is quite defective. This proof consists in ascertaining how these integrals, supposing them to exist, would behave, should we attempt to derive periodic solutions from them. It is difficult to present this matter without the assistance of algebraic formulæ; nevertheless, it may be attempted. Let there be a number of equations whose left members are formed by the product of two factors. When we pass to a periodic solution, one of these factors becomes zero. What conclusion can we draw from each of the thus modified equations? Evidently one of two things: either the remaining factor of the left member is infinite and the right member indeterminate, or it is finite and the right member a vanishing quantity. Now in case we are obliged to accept the first conclusion, were it only but once, M. Poincaré has demonstrated the non-existence of integrals; but, granting that it is proper in every case to accept the latter conclusion, the demonstration fails. Now he declines to consider the latter alternative, saying that he does not believe that any problem of dynamics, presenting itself naturally, occurs where the right members of the mentioned equations would all vanish. But it should be borne in mind that, while they do not vanish in the general equations, the adjustment of the values of the linear parameters required by the passage to a periodic solution may bring about their vanishing. Thus, in the lunar theory, a periodic solution is brought about by making $e = 0$, $e' = 0$, and $\gamma = 0$, the result is the vanishing of every coefficient having any of these quantities as a factor.

M. Poincaré appeals in another place to the fact that the Lindstedt series, if convergent, would establish the non-existence of asymptotic solutions. But this observation is irrelevant for the reason that the domains of the two things are quite distinct.

In any case where Lindstedt series are applicable there are no asymptotic solutions, and, where there are asymptotic solutions, Lindstedt's series would be illusory.

We owe much to M. Poincaré for having commenced the attack on this class of questions. But the mist which overhangs them is not altogether dispelled; there is room for further investigation.

KRONECKER'S LINEAR RELATION AMONG MINORS OF A SYMMETRIC DETERMINANT.

BY PROFESSOR HENRY S. WHITE.

AMONG the minors of any determinant there exist well-known identical relations; those of lowest order, the quadratic relations, being readily obtained by the expansion of a determinant in which at least one pair of rows or of columns are identical. If, however, the original determinant is symmetrical, there are identities of a lower order than the quadratic, the linear identities first formally noticed by Kronecker in 1882.* These linear relations, published with no hint as to the manner of their discovery, are suggestive of a certain formula in such constant use as to have become a commonplace in the transformations of the Theory of Invariants of linear substitutions. The latter formula, however, relates to products of two determinant-factors, while Kronecker's is linear; but the latter uses double indices for the constituents, and herein lies the resemblance. By virtue of the ordinary process of multiplication of two determinants, Kronecker's theorem is easily proved to be a consequence from the other identity. Both are equally general, hence it seems likely that the earlier may have been the source of the later. This theory I will develop inductively, using for the sake of brevity determinants of three rows, and obtaining a typical linear relation among three-rowed minors of a six-rowed symmetric determinant.

Form an array of three rows of six constituents each:

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{matrix}$$

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* *Sitzungsberichte der Berliner Akademie*, 1882, p. 824. See also C. Runge: Die linearen Relationen zwischen den verschiedenen Subdeterminanten symmetrischer Systeme. *Jour. für r. u. a. Math.*, vol. 93 (1882), pp. 319–327.

Denote the determinant of any three columns in this array by the three indices in proper order, as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1\,2\,3).$$

The identity employed in transforming invariants will then be written as follows:

(1) $(1\,2\,3)(4\,5\,6) \equiv (1\,2\,4)(3\,5\,6) + (1\,2\,5)(4\,3\,6) + (1\,2\,6)(4\,5\,3).$

The correctness of this is seen upon developing in minors of the first three, and of the last three rows the vanishing determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ 0 & 0 & a_3 & a_4 & a_5 & a_6 \\ 0 & 0 & b_3 & b_4 & b_5 & b_6 \\ 0 & 0 & c_3 & c_4 & c_5 & c_6 \end{vmatrix} \equiv 0.$$

Multiplying the determinants of each pair in (1), column by column, we have in each product-determinant constituents all of the same form, which can be abbreviated thus:

$$a_1a_4 + b_1b_4 + c_1c_4 = 1, 4,$$

or

$$a_ia_k + b_ib_k + c_ic_k = i, k.$$

Identity (1) will thus become:

(2)
$$\begin{vmatrix} 1,4 & 2,4 & 3,4 \\ 1,5 & 2,5 & 3,5 \\ 1,6 & 2,6 & 3,6 \end{vmatrix} \equiv \begin{vmatrix} 1,3 & 2,3 & 4,3 \\ 1,5 & 2,5 & 4,5 \\ 1,6 & 2,6 & 4,6 \end{vmatrix} + \begin{vmatrix} 1,4 & 2,4 & 5,4 \\ 1,3 & 2,3 & 5,3 \\ 1,6 & 2,6 & 5,6 \end{vmatrix} + \begin{vmatrix} 1,4 & 2,4 & 6,4 \\ 1,5 & 2,5 & 6,5 \\ 1,3 & 2,3 & 6,3 \end{vmatrix}$$

We may observe, from the definition of the symbol, that $i, k = k, i$. Accordingly if the 36 quantities i, k be arranged as a six-rowed determinant, in the order denoted by their double indices, that determinant will be symmetric; that is, if

$$A_{ik} = A_{ki} = (a_ia_k + b_ib_k + c_ic_k),$$

then, (3)
$$D_6 = |A_{ik}| (i, k = 1, 2, 3, 4, 5, 6)$$

will be a symmetric determinant, of which the determinants in (2) above will be third minors.

The identity (2) *is a Kronecker relation among three-rowed minors of the six-rowed symmetric determinant* D_6.

Such relations are thus derived for certain symmetric six-rowed determinants. D_6, however, is of a highly specialized type; it is the discriminant of the sum of three 6-ary squares. Is the relation (2), established for this special type, valid for all symmetric determinants of six rows? It is; for it involves no constituent from the principal diagonal, so that the 18 parameters of our 6×3 array are available for representing the 15 constituents $\left(\frac{6 \cdot 5}{1 \cdot 2} = 15\right)$, lying outside the principal diagonal of any given symmetric 6-rowed determinant.

The same consideration can be relied upon in adapting this proof to any Kronecker relation among m-rowed minors of a $2m$-rowed symmetric determinant, since always

$$\frac{2m(2m-1)}{2} < 2m^2.$$

For convenience of reference, I subjoin the general form of this relation as enunciated by Kronecker, *loc. cit.*:

$$|a_{gh}| = \sum_{r} |a_{ik}|,$$

$$\begin{pmatrix} g=1, 2, \cdots, m; & h=m+1, m+2, \cdots, 2m \\ i=1, 2, \cdots, m-1, r; & k=m+1, m+2, \cdots, r-1, m, r+1 \cdots 2m \end{pmatrix}.$$

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ON THE LISTS OF ALL THE SUBSTITUTION GROUPS THAT CAN BE FORMED WITH A GIVEN NUMBER OF ELEMENTS.

BY DR. G. A. MILLER.

T. P. KIRKMAN published in 1863 the first extensive list of all the transitive substitution groups that can be formed with a given number of letters. A number of interesting facts are associated with this list. Before entering upon its discussion we shall give a brief account of the more important direct steps towards the formation of such lists.

Paolo Ruffini published a work* in 1799 in which we do

* The complete title of this work in two volumes is, "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto, di Paolo Ruffini,*" Bologna, 1799.

not only find groups divided into transitive,* intransitive, primitive and non-primitive, but also an attempt to give all the orders of the groups that can be formed with five elements. This work is especially interesting because it contains some concepts which have been attributed to Cauchy for a long period of years.†

In 1845 Augustin Cauchy published in *Comptes Rendus*, vol. 21, pp. 1363–1369, lists of all the possible orders of the groups that can be formed with two, three, four, five, and six letters. He determines the orders of the intransitive, non-primitive and primitive groups separately. His lists contain a few orders for which no group of the given class exists, and the list of the orders of the non-primitive groups of degree six does not contain 36. There are no other omissions. Some theorems relating to group construction precede these lists.

In 1850 J. A. Serret published in *Liouville's Journal* complete lists of all the substitution groups that can be formed with four and with five letters. No special notation is employed, but the groups are written in ordinary language in the summaries on pages 52, 53 and 65–67. The facts that these lists are complete and that they contain the groups themselves instead of their orders only are evidences of a significant step forward even if the author confined himself to a very small number of letters.

Emile Mathieu continued the lists which Cauchy began, by enumerating all the possible orders of groups which contain seven and eight letters. These lists are given in *Comptes Rendus*, vol. 46, pp. 1048 and 1208 respectively.

Kirkman's List.

This is found in "*Proceedings of the Literary and Philosophical Society of Manchester*" (1863), vol. 3, pp. 133–152.‡ A table of corrections containing ten additional groups and some changes of the notation of the groups in the list is given by the same author on p. 172, vol. 4, of the same journal.

We reproduce below, the first part of this list, through degree

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* Cf. JORDAN, *Traité des substitutions*, preface VII.; MATHIEU, *Comptes Rendus*, 46, pp. 1048 and 1049.

† Cf. H. BURKHARDT, *Schlömilch's Zeitschrift*, 1892, supplement, p. 159. In the same article is found a discussion of the part of Ruffini's work which relates to substitution groups, pp. 133–137. We refer to this article for a more complete account of Ruffini's work.

‡ The heading of the article in which this list is given is, "The complete theory of groups, being the solution of the mathematical prize question of the French Academy for 1860. By the Rev. Thos. P. Kirkman, M.A., F.R.S., and honorary member of the literary and philosophical societies of Manchester and Liverpool."

four, for the purpose of adding clearness to the following remarks. It may be observed that the group of two letters is not given in the list.

$3\cdot 1=1+2_2$, $Q=1$; $3\cdot 2=1+2_3+3_{21}$, $Q=1$,
$4\cdot 1=1+2_4+1_{2^2}$, $Q=3$; $4\cdot 2=1+2_4+3_{2^2}+2_{21^2}$, $Q=3$,
$4\cdot 3=1+8_{31}+3_{2^2}$, $Q=1$; $4\cdot 6=1+8_{31}+3_{2^2}+6_4+6_{21^2}$, $Q=1$.*

Explanations of the notation. The first members of these equations give the orders of the groups. The second members give the enumeration of the different types of substitutions found in the groups. The subscripts indicate the degree of the cycle or cycles that compose the substitutions of the given type. Q stands for the number of conjugates of the group in the corresponding symmetric group.

The list and the table of corrections contain all the transitive groups through degree seven with exception of the group of degree two mentioned above. The following six groups of degree eight are omitted.

Order.	Number.	Group.
16	1	$\left\{\begin{matrix}(ae\,.\,bf\,.\,cg\,.\,dh)B'(abcd\,.\,efgh) \text{ or} \\ (aebf\,.\,cdgh)\,\text{cyc}\,(ac\,.\,bd)(ef\,.\,gh),\end{matrix}\right\}$
24	1	$(abcd\,.\,efgh)\,\text{pos}\,(ae\,.\,bg\,.\,cf\,.\,dh)$,
96	1	$(ae\,.\,bf\,.\,cg\,.\,dh)\{(abcd)_{12}(efgh)_{12}\}$ ter,
	2	$(ae\,.\,bg\,.\,cf\,.\,dh)\{(abcd)_{12}(efgh)_{12}\}$ ter,
192	1	$\{(abcd)\,\text{all}\,(efgh)\,\text{all}\}\,\text{sex}\,(ae\,.\,bf\,.\,cg\,.\,dh)$,
576	1	$\{(abcd)\,\text{all}\,(efgh)\,\text{all}\}\,\text{pos}\,(aebf\,.\,cg\,.\,dh)$.

The two given groups of order 16 are conformal † and they have the same number of conjugates in the symmetric group of degree eight. It is, therefore, impossible to determine which of these is omitted, since Kirkman's notation for the two is exactly the same. The fact that his notation involves such uncertainty is an objection to it. A more serious objection is furnished by the fact that the notation does not enable one to write directly the substitutions of the group.

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* The list does not contain the four-group. This is given in the table of corrections together with two missing groups of degree six, viz., $(abcdef)_{12_2}$ and $\pm(abcdef)_{24}$. The list is complete for the degrees three, five, and seven.

† For brevity we use the term "conformal groups" for non-conjugate groups, all of whose substitutions are of the same type. Kirkman does not give any conformal groups in the list itself. The table of corrections, however, contains a group which is conformal to one in the list, viz., a G_{32}. Cf. Kirkman, *Mathematics from Educational Times*, vol. 38, pp. 80, 81.

It appears quite likely that Kirkman regarded the two given groups of order 16 as conjugate without going through the full proof. Two other omissions may be accounted for in almost the same way. Kirkman's group of order 96, and the first group of this order given above, as well as the two positive transitive groups of order 192, are also conformal. In each of these cases the omitted group has 210 conjugates in the symmetric group of degree eight, while the given group has only 105, so that it is easy to determine which of the conformal groups are in the list.

The other three omitted groups cannot be accounted for in the same way, since Kirkman's notation for these groups would be quite distinct from that used for other groups in the list.

It may be observed that while Kirkman omitted six transitive groups of degree eight there are twenty-four such omissions in Cayley's list,* which was published about thirty years later.

The six groups, which are not given in Kirkman's list and table of corrections, have been published as follows:

Three, those of orders 16 and 96, in Cayley's list mentioned above. The other three were first published in this journal; those of orders 192 and 576,† May, 1893, and the one of order 24, April, 1894.

For degree nine Kirkman again omitted six groups, viz.:

Order.	Number.	Group.
54	1	$(abc.def)\,\mathrm{cyc}\,(abc.ghi)\,\mathrm{cyc}\,(adg.bfi.ceh) \times (ad.be.cf.gh),$
	2	$\left\{ \begin{array}{l} [\{(abc)\mathrm{all}(def)\mathrm{all}\}\mathrm{pos}(ghi)\mathrm{all}]\,\mathrm{sex}(adg.bfi.ceh) \\ \quad \text{or } (abc.def)\mathrm{cyc}(abc.ghi)\mathrm{cyc}(adg.bfi.ceh)\mathrm{cyc} \\ \quad \times (ad.bf.ce), \end{array} \right\}$
72	1	$(abcd.efgh)\,\mathrm{cyc}\,(ac.ef.gh)(aci.bgh.dfe.)\,\mathrm{cyc},$
108	1	$[\{(abc)\,\mathrm{all}\,(def)\,\mathrm{all}\}\,\mathrm{pos}\,(ghi)\,\mathrm{all}]\,\mathrm{sex} \times (adg.bfi.ceh)\,\mathrm{cyc}\,(ad.be.cf.gh),$
162	1	$\{(abc)\,\mathrm{pos}\,(def)\,\mathrm{pos}\,(ghi)\,\mathrm{pos} + (abc)\,\mathrm{neg}\,(def)\,\mathrm{neg}\,(ghi)\,\mathrm{neg}\}(adg.beh.cfi)\,\mathrm{cyc},$
648	1	$\{(abc)\,\mathrm{all}\,(def)\,\mathrm{all}\,(ghi)\,\mathrm{all}\}\mathrm{pos} \times (adg.beh.cfi)\,\mathrm{cyc}\,(ad.be.cf).$

It is impossible to decide which one of the two groups 54_2 is given by Kirkman, for they are conformal, and one has 1120 while the other has 3360 conjugates in the symmetric group of

* *Quarterly Journal of Mathematics*, vol. 25, pp. 137–155.

† These two are given in Professor Cole's "Note on the substitution groups of six, seven, and eight letters." This "note" is also found in *Quarterly Journal of Mathematics*, vol. 26, pp. 372–376.

degree nine. Kirkman gives 1680 as the value of Q, which is certainly erroneous.

It may be well to state here that numerous errors occur in Kirkman's notation for other groups, especially in the higher orders. He never gives any more groups of a given order than actually exist. It therefore seems quite likely that the errors in notation are due to insufficient study of the groups under consideration. This view is, perhaps, supported by the author's statement in the introduction to the list, viz., "If any (transitive groups whose degree is less than 11) have escaped me, it is the fault, not of my method, but of my carelessness." For degree ten the list is so inaccurate as to be of little value.

In 1872, Camille Jordan gave an enumeration of the numbers of primitive groups for each degree through degree seventeen in *Comptes Rendus*, 75, p. 1757. Only two omissions have been published, one of degree * nine and the other of degree twelve.†

About twenty years later ‡ Askwith, who apparently was unacquainted with Serret's list,§ published two articles in the *Quarterly Journal of Mathematics* ‖ in which he endeavored to give all the groups for each degree from three to nine, together with the methods by which they may be found. He referred only to Serret's work,¶ although two more comprehensive treatises ** had been published.

In the following number of the same journal, Professor Cayley gave the results which Askwith had obtained, together with some additions, in a condensed and perspicuous form.†† In the introduction to this list he explains his notation and calls attention to some very important principles in group construction. The list is very inaccurate. Twenty intransitive and twenty-eight transitive groups were omitted.

Since Cayley's notation is far superior to Kirkman's, and since the corrections to Cayley's list are published in the same notation as the list itself, it may be of especial interest to give an outline of these corrections.

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* This journal, vol. 3, p. 245.

† *Ibid.* vol. 1, 2d series, p. 255.

‡ The most important work in the line of our subject, which was published in this period, is that of VERONESE, *Annali di Mathematica* (2), vol. 11, pp. 93–233. This contains lists of the groups of degrees 2, 3, 4, 5, and 6, on pp. 153, 157, 158, 167, 168, 177–190, 232, 233. Both the transitive and the intransitive groups are enumerated.

§ Cf. *Quarterly Journal of Mathematics*, vol. 24, p. 111.

‖ Vol. 24, pp. 111–167 and 263–331.

¶ *Ibid.* pp. 111, 144, 270.

** JORDAN, *Traité des Substitutions*, 1870, and NETTO, *Substitutionentheorie*, 1882. CAUCHY'S work, *Exercices d'analyse*, vol. 3, pp. 151–252 (1844), is not quite as comprehensive as Serret's.

†† *Quarterly Journal of Mathematics*, vol. 25, pp. 71–88 and 137–155.

In the "Note on the substitution groups of six, seven, and eight letters,"* Professor Cole gave all the missing groups except two. Some errors in this note are corrected by its author in the *Quarterly Journal*.† It remains to add that in the enumeration at the end of the note, 199 and 48 should be replaced respectively by 200 and 50. The two missing groups were published separately in this journal.‡

It is very probable that the groups through degree eight have been completely determined. The following enumeration may therefore be accepted with a great deal of confidence as a permanent one.

Number of letters	2	3	4	5	6	7	8
Number of groups	1	2	7	8	37	40	200
Transitive groups	1	2	5	5	16	7	50

Through degree five Serret's list of 1850 was complete, as stated before.

The work begun by Askwith in the two articles mentioned above is continued in vol. 26 of the same journal. His attempt to determine all the transitive groups of degree nine was not wholly successful.§

All the groups of degree nine‖ and the transitives of degrees ten¶ and eleven have recently been published by Professor Cole, and the intransitives of degree ten** by the author. From the preceding it will be seen that these enumerations according to degree have been pursued to the following extent. All the possible groups through degree ten, the transitives through degree eleven,†† and the primitives through degree seventeen have been determined.

It should not be inferred that the enumerations according to degree are the only ones which have received attention. In addition to these, there are enumerations of the primitive groups according to class,‡‡ of simple groups

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* This journal, May, 1893; *Quarterly Journal of Mathematics*, vol. 26, pp. 372–376.

† Vol. 27, p. 50.

‡ Vol. 3, pp. 168 and 242–244. Professor Moore informed me that this group had been determined in practically the same manner by NOETHER, *Mathematische Annalen*, vol. 15, pp. 90–95.

§ Cf. this journal, vol. 2, p. 250.

‖ *Quarterly Journal of Mathematics*, vol. 26, pp. 376–388. Cf. *Ibid.* vol. 27, p. 102, foot-note.

¶ *Ibid.* vol. 27, pp. 39–50. Cf. this journal, vol. 1, 2d series, pp. 67–72.

** *Quarterly Journal of Mathematics*, vol. 27, pp. 99–118.

†† The transitives of degree twelve will appear in vol. 28, *Quarterly Journal*.

‡‡ JORDAN, *Comptes Rendus*, vol. 75, p. 1757; NETTO, *Theory of Substitutions* (COLE's edition), pp. 133–138.

according to order,* of groups whose orders are composed of given kinds of factors,† of regular groups according to degree,‡ etc.

Methods of Group Construction.

Most of the lists which have been mentioned are preceded by brief explanations of the methods employed by the authors, and references are generally given to works where more complete explanations may be found. Among the separate articles on this subject, special attention should be called to the two following: "On the construction of intransitive groups"§ by Professor Bolza, and "Construction of composite groups" ‖ by Professor Hölder.

The methods of finding all the groups of a given degree may be divided into two great classes: (1) those by means of which we prove the existence of groups and find their distinguishing features, and (2) those by means of which we prove the non-existence of given types of groups in certain regions. The method of the second class employed in the following examples has extensive application in the construction of such non-primitive groups.

In constructing the non-primitive groups of degree ten, it is necessary to inquire whether there is a non-primitive group with five systems of two letters which is simply isomorphic to (*abcde*) pos. If we interchange the systems only according to the substitutions of the given group in five letters and preserve the same order of the letters of the systems throughout, we evidently obtain (*abcde . fghij*) pos. The average number of letters in this group is eight,¶ and the average number in the required group is nine.¶ If such a group exists, it must therefore be possible to solve the Diophantine equation,

$$20a + 15b = 60 \ (a = 0, 2, 4; \ b = 0, 2).$$

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* HÖLDER, *Mathematische Annalen*, vol. 40, pp. 55–89; COLE, *American Journal*, vol. 14, pp. 378–388, and vol. 15, pp. 303–316; FROBENIUS, *Berliner Mittheilungen*, 1893, pp. 221–229; *Sitzungsberichte*, 337–345.

† NETTO, *Theory of Substitutions* (COLE'S edition), pp. 145–149; YOUNG, *American Journal*, vol. 15, pp. 124–179; COLE and GLOVER, *ibid.* pp. 191–221; HÖLDER, *Annalen*, vol. 43, pp. 301–413, and *Göttinger Nachrichten*, 1895, pp. 211–229; SYLOW, *Acta Mathematica*, vol. 11, pp. 201–256; BURNSIDE, *Proceedings of the London Mathematical Society*, vol. 25, pp. 9–18.

‡ KEMPE, *Philosophical Transactions*, vol. 177, pp. 37–43; CAYLEY, *American Journal*, vol. 11, pp. 139–157; LEVAVASSEUR, *Comptes Rendus*, vol. 120, pp. 822–825 and 899–903; vol. 121, pp. 238–240.

§ *American Journal of Mathematics*, vol. 11, pp. 195–214.

‖ *Mathematische Annalen*, "Bildung zusammengesetzter Gruppen," pp. 321–422. A series of articles on elementary methods of group construction are now being published in the *American Mathematical Monthly*.

¶ FROBENIUS, *Crelle*, vol. 101, p. 287. Cf. "A simple proof of an elementary theorem in substitutions," etc. This journal, current vol., p. 75.

This equation depends directly upon the fact that the substitutions of the non-primitive group must interchange all the systems represented by the letters of the corresponding substitution of $(abcde)$ pos, and leave the remaining systems unchanged. Since the given Diophantine equation has clearly no solution, no non-primitive group of the required type exists.

Similarly there can be no non-primitive group of degree twelve whose six systems of two letters correspond to the elements of $(abcdef)$ pos and whose base is unity unless the following equation can be solved,

$$40a + 45b + 144c = 360$$

$$(a = 0, 2, 4, 6;\ b = 0, 2, 4;\ c = 0, 2).$$

This has evidently no solution. It is not necessary to examine the corresponding symmetric groups; for, if it is possible that a non-primitive group in the given number of systems is simply isomorphic to either of them, its self-conjugate subgroup corresponding to the alternating group must be intransitive. This could, therefore, be regarded as the base, and its systems of intransitivity as the systems of the required non-primitive group.

It remains to notice a certain dual interpretation of the elements of a non-regular primitive group. If we represent the subgroups which do not contain $a, b, \cdots$ by $A, B, \cdots$, then will every substitution of the group $(abc.de\cdots)$ interchange these subgroups according to an identical substitution in the capital letters $(ABC.DE\cdots)$. We may, therefore, think of the substitutions as interchanging the elements or as interchanging these subgroups. This dual interpretation is not always so interesting in the other groups, since there need not then be a 1, 1 correspondence between the elements and the capital letters.*

Whenever all the capital letters stand for subgroups whose degree is one less than the degree of the group, such a 1, 1 correspondence must exist even if the group is intransitive or non-primitive.

LEIPZIG, *October*, 1895.

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* Cf. MOORE, this journal, vol. 1, 2d series, pp. 61, 62.

ON CAUCHY'S THEOREM CONCERNING COMPLEX INTEGRALS.

BY PROFESSOR MAXIME BÔCHER.

THE following arrangement of the proof of Cauchy's theorem, that the integral of a holomorphic function in a simply connected region is a function of the limits of integration, but not of the path, is more elementary than the proofs ordinarily met with. Like the proofs given by Goursat* and Jordan,† it avoids the use of double integrals or the calculus of variations, and it has the advantage over these proofs not merely of brevity, but also of avoiding more or less complicated considerations of limits. The materials from which the following proof is built up are so familiar to all students of the theory of functions,‡ and the consequent probability that this method of putting them together is not unknown, is so great, that I should hardly have ventured to publish it, had it not seemed to possess peculiar pedagogical advantages. These depend, apart from its elementary character, on the fact that almost all the steps involved establish theorems, or illustrate methods, with which the student must become familiar sooner or later.

The theory of integrals of a function of a complex variable depends, as is well known, on the theory of integrals of the form

$$I = \int_{(a,b)}^{(x,y)} (Pdx + Qdy),$$

in which P and Q are real functions of the two real variables x, y. We will begin with the theorem

S being any region in the xy-plane in which P and Q are single-valued and have continuous first partial derivatives, a necessary and sufficient condition that the integral I should depend merely on the limits of integration is the existence in S of a single-valued function $\phi(x, y)$, for which $\frac{\partial \phi}{\partial x} = P$, $\frac{\partial \phi}{\partial y} = Q$.

For, if such a ϕ exists, $Pdx + Qdy$ is the complete differential of ϕ, and the integral I is the limit of the sum of the increments of ϕ, which we get in going along the path of integration from (a, b) to (x, y). But the sum of these increments is $\phi(x, y) - \phi(a, b)$, so that I is independent of the path of integration. Conversely, if I is independent of the path, it

* See HARKNESS and MORLEY'S *Theory of Functions*, p. 164.

† *Cours d'Analyse* (2d ed.), vol. 1, p. 185.

‡ See, for instance, PICARD, *Traité d'Analyse*, vol. 1, pp. 81–83, and BOUSSINESQ, *Cours d'Analyse infinitésimal*, vol. 2, pp. 6–12.

will be itself (the lower limit (a, b) being regarded as constant) a function $\phi(x, y)$ of the sort we want; for, in the first place, I is now single-valued in S by hypothesis. Its increment when x alone changes is

$$\Delta_x I = \int_{x,y}^{(x+\Delta x, y)} (Pdx + Qdy).$$

In this integral the path of integration may be taken as a straight line, so that y is constant. We get then

$$\Delta_x I = \int_x^{x+\Delta x} P(x, y)dx = \Delta x \cdot P(x + \theta\Delta x, y),$$

where $0 \leqq \theta \leqq 1$. Dividing through by Δx, and taking the limit as Δx approaches zero, we get $\frac{\partial I}{\partial x} = P(x, y)$. Similarly, we find $\frac{\partial I}{\partial y} = Q(x, y)$.

Obviously, then, a necessary condition that I should be independent of the path is that at every point of S

$$(A) \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

since both sides of this equation will be $\frac{\partial^2 \phi}{\partial x \partial y}$. It remains to show that this is also a sufficient condition, provided that the region included between the paths of integration belongs wholly to the region S, as would necessarily be the case if S were simply connected. We first prove the following:

Lemma. — Equation (A) is a sufficient as well as a necessary condition for rectangular regions S whose sides are parallel to the axes of x and y.

To establish this lemma, we must show that if equation (A) holds throughout our rectangle, a single-valued function ϕ exists there, such that $\frac{\partial \phi}{\partial x} = P$, $\frac{\partial \phi}{\partial y} = Q$. Such a ϕ we get here again in the integral I itself, provided that, in order to ensure that I shall be a single-valued function of x, y, we agree to make the path of integration consist of two straight lines, — from (a, b) to (x, b) and from (x, b) to (x, y). Owing to the restricted shape of S, these two lines will always lie wholly in S. We have then

$$\phi = \int_a^x P(x, b)dx + \int_b^y Q(x, y)dy,$$

$$\frac{\partial \phi}{\partial y} = Q(x, y),$$

$$\frac{\partial \phi}{\partial x} = P(x, b) + \int_b^y \frac{\partial Q(x, y)}{\partial x} dy = P(x, b) + \int_b^y \frac{\partial P(x, y)}{\partial y} dy = P(x, y).$$

From the lemma just proved, it follows at once that equation (A) is the necessary and sufficient condition that the integral I should vanish when taken around any closed path which lies within or upon the boundary of a rectangle of the sort we have just considered.

We will now state the final proposition we have to prove in the following form:

Whatever shape the region S may have, the sufficient as well as the necessary condition that the integral I taken in the positive direction around the complete boundary of any region S' lying actually within S should be zero, is equation (A).*

To prove this, let us divide up that portion of S where S' lies into rectangles by lines parallel to the axes of x and y, and take these rectangles so small† that those rectangles which lie partly within and partly without S' shall lie wholly within S. Then S' will be divided into a number of pieces, some of which are rectangles, while the others are parts of rectangles. The integral taken in the positive direction around the complete boundary of S' is obviously equal to the sum of the integrals taken in the positive direction around the boundary of each of the small pieces into which we have divided S'. But each of these last integrals vanishes if equation (A) holds throughout S, since each of the paths lies within or on the boundary of a rectangle of the sort considered in the preceding lemma, throughout which equation (A) holds. The integral taken around the complete boundary of S' will therefore vanish, if (A) holds throughout S.

Bearing in mind that

$$\int (u+vi)(dx+idy)=\int (udx-vdy)+i\int (vdx+udy),$$

we infer at once from the proposition just proved the following proposition:

u and v being single valued real functions of the real variables x, y in a region S of the xy-plane, and having there continuous first partial derivatives, the necessary and sufficient condition that $\int (u+vi)(dx+idy)$ taken around the boundary of a region S' which lies within S should be zero is:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

* That is, the boundary of S' must not coincide with that of S at any point.

† It should be carefully noticed that these rectangles are not "infinitesimal," *i.e.* that we are not going to allow them to decrease indefinitely.

It might be objected that in the above proof we assume that the region S' around whose boundary we integrate lies within another region S, throughout which the conditions of continuity and the equation (A) are satisfied. In the first place, however, this will be true in all cases to which we ordinarily apply the theorem;* and in the second place the proof can easily be so modified as to obviate this difficulty, at least for all ordinary shapes of the boundary. All we should have to do would be by a slight extension of the method to prove the lemma given above, not for rectangles with sides parallel to the axes of x and y, but for regions bounded by three sides of such a rectangle, and on the fourth side by a curve which is cut by no line parallel to the two parallel straight sides in more than one point. It will clearly be possible, in any ordinary case, to take the rectangles into which we cut up the region S' so small that the pieces of rectangles which occur near the boundary of S' shall be of this nature.

HARVARD UNIVERSITY, *December*, 1895.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, January 25, at three o'clock, the President, Dr. HILL, in the chair. There were seventeen members present. One nomination for membership was received. The report of the auditing committee, appointed at the preceding meeting to examine the Treasurer's accounts, was presented and accepted. The following papers were read:

(1) Professor HENRY S. WHITE: "Kronecker's linear relation among the minors of a symmetric determinant."

(2) Professor H. TABER: "On certain sub-groups of the general projective group."

In the absence of Professor White his paper was read by Mr. Ling.

THE following mathematical courses are offered in the University of Leipzig for the summer semester of the present year: — Professor Scheibner: Theory of numbers; — Professor Neumann: Selected chapters in mathematical physics; — Professor Lie: Theory of groups (continuous transformation groups); — Professor Mayer: Differential equations of dynam-

* The only case when it would not be true would be when the path of integration meets a natural boundary of the function we are integrating.

ics; — Professor Engel: Ordinary differential equations; — Dr. Scheffers: (1) Theory of surfaces; (2) Introduction to higher mathematics; — Dr. Hausdorff: (1) Analytic geometry; (2) Introduction to the mathematical theory of assurance. Owing to the illness of Professor Bruns, the astronomical courses have not yet been announced.

The number of mathematical students at Leipzig (not including those studying mathematics as a minor) is at present 42; last summer it was 37, the preceding winter 25.

THE Royal Academy of Science and Letters of Denmark offers a gold medal to the author of a memoir containing a notable contribution to the solution of the following problem: *To indicate the necessary and sufficient criterions to decide whether an algebraic equation with given numerical coefficients belongs, or does not belong, to the class of Abelian equations.*

In presenting the question, it is stated that notwithstanding the great progress of algebra in modern times, with the exception of equations soluble by square roots, the problem to determine whether or not a given algebraic equation of unknown origin admits of solution, is to-day substantially where it was at the beginning of the century. However, it can hardly be doubted that the evolution of algebra during this epoch should render some advancement possible and furnish, in some cases at least, new methods of determining whether such an equation is of the class soluble through the extraction of roots.

Competitors should send their manuscripts before the end of October, 1896, to the Secretary of the Academy, Professor G. H. Zeuthen, of the University of Copenhagen. Decision will be rendered during the month of February following, after which authors may withdraw their memoirs.

REPORT OF THE TREASURER FOR THE YEAR 1895.

The report which I have the honor to submit to you exhibits the receipts and disbursements for the year ending December 27, 1895.

In the receipts are included: 1st, the amounts received for dues and initiation fees for the year 1895; 2d, the amounts received for dues of previous years; 3d, the amounts received for dues of the year 1896; 4th, the amounts received from subscriptions to and sales of the Bulletin; and 5th, the amounts received from subscriptions to fund for publication of papers presented at the Mathematical Congress of the Columbian Exposition.

The total receipts of the Society during the past year, as shown by the following itemized statement, amount to $1423.26. The disbursements for the same period amount to $862.63. The itemized vouchers for these disbursements are herewith submitted in Exhibit A.

The Society's present cash assets, including in these a total of $192.50 of subscriptions to the publication fund mentioned above, on deposit in the Fifth Avenue Bank of New York, are $1677.30. The net increase in assets during the past year is $560.63.

R. S. WOODWARD, *Treasurer.*

NEW YORK, *December* 27, 1895.

The Treasurer in Account with the American Mathematical Society.

Dr.		*Cr.*	
1895.		1895.	
Dec. 27.		Dec. 27.	
To balance on hand Dec. 27, 1894	$1116.67	By cash paid Macmillan & Co. for publication of 8 numbers of Bulletin and extra copies thereof	$556.50
To cash received for dues and initiation fees for the year ending Dec. 27, 1895	1156.26	By cash paid William R. Jenkins for stamped envelopes; for printing notices of meetings, circulars, ballots, and blanks; and for stationery	168.56
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To cash received from subscriptions to and sales of the Bulletin	119.50		
To cash received from subscriptions to fund for publication of papers of Mathematical Congress of Columbian Exposition	32.50		
	$2539.93		$2539.93

REPORT OF AUDITING COMMITTEE.

The Committee finds that all sums entered as receipts in the Treasurer's books are properly accounted for by the balance remaining to the Society's credit in the bank and by proper vouchers for expenditures.

F. N. COLE,
J. B. CHITTENDEN,
POMEROY LADUE.

NEW YORK, *January* 25, 1896.

NEW PUBLICATIONS.

HIGHER MATHEMATICS.

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. (In 3 Bänden.) Band III: Vom Jahre 1668 bis zum Jahre 1759. (In 3 Abteilungen.) Abteilung II: Die Zeit von 1700 bis 1726. Leipzig, Teubner, 1896. 8vo. pp. 253–472. Mk. 6.00

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D'OVIDIO (E.). Geometria analitica. Torino, Bocca, 1896. 8vo. 16 and 443 pp. Fr. 10.00

FAVARO (A.). Lezioni di geometria proiettiva. 3a edizione, riveduta ed aumentata. Padova, Sacchetti, 1895. 8vo. 8 and 260 pp. Fr. 5.00

GUYOT (E.). Simple exposé des relations à établir entre les équations et leurs racines. Paris, May et Motteroz, 1895. 8vo. 85 pp.

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SAUVAGE (L.). Théorie générale des systèmes d'équations différentielles linéaires et homogènes. Paris, Gauthier-Villars, 1895. 4to. 184 pp. Fr. 6.00

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